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A qualocation method for Burgers' equation

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Abstract

In this paper, a qualocation method for the one-dimensional Burgers' equation is proposed. A semidiscrete scheme along with optimal error estimates is discussed. Results of a numerical experiment performed support the theoretical results.

MSC: 65M70; 65M15

Keywords: Qualocation method; Burgers' equation; Optimal error estimates; Gauss quadrature rule; Order of convergence

1. Introduction

In this paper, we consider a qualocation method for Burgers' equation given by

$$u_t - \nu u_{xx} + uu_x = 0, \quad x \in I = (0, 1), \ t \in (0, T], \tag{1.1}$$

with the initial and boundary conditions

$$u(x, 0) = u_0(x), \quad x \in (0, 1), \tag{1.2}$$
$$u(0, t) = u(1, t) = 0, \quad t > 0, \tag{1.3}$$

where the positive number $\nu = 1/Re$ is the coefficient of kinematic viscosity, Re denotes the Reynolds number and u_0 is a given function. Burgers' equation is a one-dimensional version of the Navier–Stokes equation. It is widely used as a simplified model for turbulence, boundary layer behaviour, shock wave formation, convection dominated diffusion phenomena, acoustic attenuation in fog and continuum traffic simulation.

Historically, Burgers' equation was first introduced in [3] who gave its steady state solution. It was then discussed in [6,7] after whom the equation was named, as a simplified model for turbulence. This equation was solved analytically for restricted values of initial conditions independently in [14,10]. Benton and Platzman [4] surveyed the exact solution of the one-dimensional Burgers' equation.

For existence of a unique global solution, we refer to Smoller [22, p. 427, Theorem 21.1]. Moreover it is shown that the solution to (1.1) tends to zero uniformly in [0, 1] as $t \to \infty$ [22]. Higher regularity results for (1.1)–(1.3) can be derived by modifying the analysis given in [17, pp. 123–127] and using appropriate compatibility conditions.

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Several numerical methods like finite difference methods [13], finite element methods [8], mixed finite element technique [9], Chebyshev spectral collocation methods in [5] and collocation procedures using cubic B-splines [2] are used to derive approximate solution to Burgers' equations. Bressan and Quarteroni have also discussed optimal error estimates in weighted L^2-norm. In this paper, an attempt has been made to apply qualocation method to (1.1)–(1.3), which was introduced in [20] in 1988 for boundary integral equations on smooth curves. Subsequently, Sloan et al. [21] in 1993 extended the method to linear second order two-point boundary value problems, and derived optimal error estimates in $W^{j,p}$, $j=0,1,2$, $1\leqslant p\leqslant\infty$ norms without the quasi-uniform assumption on the finite element mesh.

A qualocation method is precisely a quadrature based modification of the collocation approximations. This method can also be thought of as a discrete Petrov–Galerkin method using a cubic spline trial space and a piecewise linear test space. Complete discretization is achieved by approximating the integrals by composite two-point Gauss quadrature rule. One practical advantage of this method over the orthogonal cubic spline collocation method [18] is that for a given partition there are only half the number of unknowns and, therefore, it reduces the size of the matrix and hence, the computational cost. Jones and Pani [15] discussed the qualocation method for a second order semilinear two-point boundary value problem. Subsequently, Pani [19] expanded the scope of this method by extending the analysis to parabolic initial and boundary value problems in one space dimension. Recently, a qualocation method is also applied to uni-dimensional single phase Stefan problem by Jones and Pani [16] and optimal error estimates are discussed.

The layout of the paper is as follows. In Section 2, the qualocation method is introduced for Burgers' equation. Optimal error estimates for the semi-discrete scheme are derived in Section 3. Finally, in Section 4, numerical implementation of the scheme is discussed and computational order of convergence is derived.

2. The qualocation method

For our subsequent use, we need the following definitions. Let $W^{m,p}(I)$, $1\leqslant p\leqslant\infty$, $m\in\mathbb{N}$ denote the standard Sobolev spaces:

$$W^{m,p}(I)=\{\psi\in L^p(I): D^j\psi\in L^p(I),\ \ j=1,2,\dots,m\}$$

with norm

$$\|\psi\|_{W^{m,p}(I)}=\left(\sum_{j=0}^{m}\|D^j u\|^p_{L^p(I)}\right)^{1/p},\quad \text{for } 1\leqslant p<\infty,$$

and for $p=\infty$

$$\|\psi\|_{W^{m,\infty}(I)}=\max_{0\leqslant j\leqslant m}\|D^j\psi\|_{L^\infty(I)},$$

where $D^j\psi$ denotes the jth derivative of ψ in the sense of distributions.

When $p=2$, we simply denote $W^{m,2}(I)$ by $H^m(I)$ with the norm $\|\cdot\|_m$. Further,

$$H_0^1(I)=\{\psi\in H^1(I):\psi(0)=\psi(1)=0\}.$$

When there is no chance of confusion, we may drop I from the definition of $W^{m,p}(I)$ and call it simply $W^{m,p}$. In the sequel, we shall also use the standard spaces $L^p(0,T;X)$ or simply call $L^p(X)$, where X is a Banach space with norm $\|\cdot\|_X$. The norm on $L^p(X)$ is denoted by $\|\psi\|_{L^p(X)}=(\int_0^T\|\psi\|^p_X)^{1/p}$, for $1\leqslant p<\infty$ and for $p=\infty$, $\|\psi\|_{L^\infty(X)}=\text{ess sup}_{0\leqslant t\leqslant T}\|\psi\|_X$.

Let $v\in H^2(I)\cap H_0^1(I)$. Multiplying (1.1) by $-v_{xx}$ and integrating with respect to x over I, we obtain the following formulation: Find $u(t)\in H_0^1(I)\cap H^2(I)$, such that for $t>0$,

$$\begin{aligned}&-(u_t,v_{xx})+\nu(u_{xx},v_{xx})=(uu_x,v_{xx})\quad\forall v\in H^2(I)\cap H_0^1(I),\\&(u(0),v)=(u_0,v)\quad\forall v\in H^2(I)\cap H_0^1(I).\end{aligned}\tag{2.1}$$

Finite dimensional approximation: For $N\geqslant 1$, let

$$\Pi_N=\{0=x_0<x_1<\cdots<x_N=1\}$$

denote a partition of [0, 1] with

$$I_k = [x_k, x_{k+1}], \quad h_k = x_{k+1} - x_k, \quad k = 0, 1, \ldots, N-1,$$

and let $h = \max_{0 \leqslant k \leqslant N-1} h_k$, such that $h \to 0$ as $N \to \infty$.

Consider the following finite-dimensional spaces:

$$S_h = \{v_h \in C^2(\bar{I}) : v_h|_{I_k} \in P_3(I_k), \quad k = 0, 1, \ldots, N-1\} \tag{2.2}$$

and

$$S_h^0 = \{v_h \in S_h, v_h(0) = v_h(1) = 0\}, \tag{2.3}$$

where $P_3(I_k)$ denotes the polynomials of degree $\leqslant 3$ defined in I_k.

The semi-discrete H^1-Galerkin method for approximating u is defined as: find $\hat{u}_h : [0, T] \to S_h^0$ which satisfies

$$-(\hat{u}_{ht}, v_{hxx}) + v(\hat{u}_{hxx}, v_{hxx}) = (\hat{u}_h \hat{u}_{hx}, v_{hxx}) \quad \forall v_h \in S_h^0, \tag{2.4}$$

with the initial condition u_0 approximated suitably by $\hat{u}_{0h} \in S_h^0$.

Since it may be difficult to evaluate the integrals in (2.4) exactly, we now apply a quadrature rule to approximate the integrals. Replace the exact inner product by the discrete approximation denoted by $\langle \cdot, \cdot \rangle$ which is defined as

$$\langle \phi, \psi \rangle = \sum_{k=0}^{N-1} \langle \phi, \psi \rangle_k \quad \forall \phi, \psi \in S_h, \tag{2.5}$$

where

$$\langle \phi, \psi \rangle_k = \frac{h_k}{2}(\phi(\xi_{k1})\psi(\xi_{k1}) + \phi(\xi_{k2})\psi(\xi_{k2}))$$

and for $k = 0, \ldots, N-1$, $j = 1, 2$

$$\xi_{kj} = \frac{1}{2}(x_k + x_{k+1}) + (-1)^j \frac{h_k}{2\sqrt{3}} \tag{2.6}$$

are obtained using the fourth order composite two-point Gauss quadrature rule [21] in the interval $[x_k, x_{k+1}]$, $k = 0, \ldots, N-1$. Now the induced discrete norm becomes $[[\psi]] = \langle \psi, \psi \rangle^{1/2}$ $\forall \psi \in S_h^0$.

The discrete H^1-Galerkin procedure is to find $u_h : [0, T] \to S_h^0$ such that

$$-\langle u_{ht}, v_{hxx} \rangle + v\langle u_{hxx}, v_{hxx} \rangle = \langle u_h u_{hx}, v_{hxx} \rangle \quad \forall v_h \in S_h^0, \tag{2.7}$$

with $u_h(0)$ as an appropriate approximation of u_0 in S_h^0 to be defined later.

As we have mentioned in the introduction, (2.7) can be thought of as a Petrov–Galerkin method with trial space S_h^0 and test space consisting of C^0-piecewise linear splines T_h where

$$T_h = \{v_h \in C^0[0, 1] : v_h|_{I_k} \in P_1(I_k), k = 0, 1, \ldots, N-1\}.$$

In this case (2.7) is written equivalently as to find $u_h(t) \in S_h^0$ satisfying

$$\langle u_{ht}, v_h \rangle - v\langle u_{hxx}, v_h \rangle + \langle u_h u_{hx}, v_h \rangle = 0 \quad \forall v_h \in T_h. \tag{2.8}$$

Eq. (2.7) or (2.8) can be interpreted as a system of nonlinear ordinary differential equations in t. An application of Picard's theorem yields existence of a unique local solution u_h in a neighbourhood of 0. As a consequence of a priori error estimates, it is possible to prove the existence of a global solution u_h in $(0, T]$, see [22].

Observe that (2.8) involves a piecewise linear test space along with a cubic spline trial space both having the same dimension whereas in the case of (2.7) both the trial and test spaces involve a polynomial of degree three. Further, a second order derivative of the element from test space is to be computed and hence for the computational purposes, (2.8) is easier to use than (2.7).

We now state the following two Lemmas for our future use.

Lemma 2.1 (*Davis and Rabinowitz [11], p. 222*). *The quadrature rule satisfies the following error bound*:

$$|\varepsilon_h(g)| = \left|\langle g, 1\rangle - \int_0^1 g\,\mathrm{d}x\right| \leqslant C \sum_{k=0}^{N-1} h_k^4 \|g^{(4)}\|_{L^1(I_k)}. \tag{2.9}$$

Lemma 2.2 (*Douglas and Dupont [12], p. 11*). *For all v and w in S_h*,

$$-\langle v, w_{xx}\rangle = (v_x, w_x) - v w_x|_0^1 + \frac{1}{1080} \sum_{k=0}^{N-1} (v_{xxx,k})(w_{xxx,k}) h_k^5, \tag{2.10}$$

where $v_{xxx,k}$ is the (*constant*) *value of the third derivative of v on I_k*.

As a consequence of Lemma 2.2, we obtain

$$-\langle v, v_{xx}\rangle \geqslant \|v_x\|^2, \quad v \in S_h^0. \tag{2.11}$$

Since the two-point Gauss quadrature rule is exact for a polynomial of degree $\leqslant 3$,

$$[\![v_{xx}]\!] = \|v_{xx}\| \quad \forall v \in S_h. \tag{2.12}$$

Moreover, from [12, p.12], we easily obtain

$$[\![v_x]\!]^2 \leqslant \|v_x\|^2 \quad \forall v \in S_h. \tag{2.13}$$

3. A priori error estimates

Since a direct comparison between u and u_h may not yield an optimal error estimate, we therefore split the error $u - u_h$ as

$$e := u - u_h = (u - \tilde{u}_h) + (\tilde{u}_h - u_h),$$

where the auxiliary function $\tilde{u}_h$ is defined by

$$\langle L(u - \tilde{u}_h), v_{hxx}\rangle = 0 \quad \forall v_h \in S_h^0, \tag{3.1}$$

and

$$L(\phi) = -v\phi_{xx} + u\phi_x + \lambda\phi.$$

Here, λ is a positive constant. Note that from [21], the problem (3.1) has a unique solution $\tilde{u}_h$ for a given u and for sufficiently small h. Write $e = u - u_h = \eta - \theta$, where $\eta = u - \tilde{u}_h$ and $\theta = u_h - \tilde{u}_h$. Following the analysis of [21], we can easily derive the following estimates of η.

Lemma 3.1 (*Pani [19]*). *There exists a positive constant C such that for sufficiently small h and for $j = 0, 1, 2$,*

$$\left\|\frac{\partial^j \eta}{\partial t^j}\right\|_{W^{i,p}} \leqslant C \sum_{l=0}^{j} h^{4-l} \left\|\frac{\partial^l u}{\partial t^l}\right\|_{W^{6,p}}, \quad i = 0, 1, \;\; p \in [1, \infty] \tag{3.2}$$

holds.

Using the definition of η, the triangle inequality and Lemma 3.1, we obtain the following bounds for $\tilde{u}_h$:

$$\|\tilde{u}_h\|_{W^{1,\infty}} \leqslant Ch^3 \|u\|_{W^{6,\infty}} + \|u\|_{W^{1,\infty}} \leqslant C. \tag{3.3}$$

Since by Lemma 3.1, the estimates of η are known, it is sufficient to prove the estimates of θ. From (1.1), (2.7) and (3.1), we obtain

$$\begin{aligned} -\langle\theta_t, v_{hxx}\rangle + v\langle\theta_{xx}, v_{hxx}\rangle = & -\langle\eta_t, v_{hxx}\rangle + \lambda\langle\eta, v_{hxx}\rangle + \langle\theta\theta_x, v_{hxx}\rangle \\ & + \langle\theta\tilde{u}_{hx}, v_{hxx}\rangle - \langle\eta\tilde{u}_{hx}, v_{hxx}\rangle + \langle\tilde{u}_h\theta_x, v_{hxx}\rangle. \end{aligned} \tag{3.4}$$

We now proceed to derive the error estimates.

Theorem 3.2. *Let u_h be a qualocation approximation of u as defined in* (2.7) *and let $u_h(x,0)=\tilde{u}_h(x,0)$. Then, there exists a constant $C>0$, independent of h such that, for sufficiently small h*

$$\|e\|_{L^\infty(L^2)} + h\|e\|_{L^\infty(H^1)} \leqslant Ch^4\{\|u\|_{L^2(W^{6,\infty})} + \|u_t\|_{L^2(W^{6,\infty})} + \|u\|_{L^\infty(W^{6,\infty})}\}, \tag{3.5}$$

and for $1\leqslant p\leqslant\infty$,

$$\|e\|_{L^\infty(L^p)} \leqslant Ch^4\{\|u\|_{L^\infty(W^{6,p})} + \|u\|_{L^2(W^{6,\infty})} + \|u_t\|_{L^2(W^{6,\infty})}\}, \tag{3.6}$$

where $e=u-u_h$.

Proof. Setting $v_h=\theta$ in (3.4), we obtain

$$\begin{aligned} -\langle\theta_t, \theta_{xx}\rangle + v\langle\theta_{xx}, \theta_{xx}\rangle &= (-\langle\eta_t, \theta_{xx}\rangle + \lambda\langle\eta, \theta_{xx}\rangle) + \langle\theta\theta_x, \theta_{xx}\rangle \\ &\quad + \langle\theta\tilde{u}_{hx}, \theta_{xx}\rangle - \langle\eta\tilde{u}_{hx}, \theta_{xx}\rangle + \langle\tilde{u}_h\theta_x, \theta_{xx}\rangle \\ &= \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4 + \mathscr{I}_5. \end{aligned} \tag{3.7}$$

For estimating the first term on the left hand side of (3.7), we use Lemma 2.2 to arrive at

$$\begin{aligned} -\langle\theta_t, \theta_{xx}\rangle &= (\theta_{tx}, \theta_x) + \frac{1}{1080}\sum_{k=0}^{N-1}(\theta_{txxx,k})(\theta_{xxx,k})h_k^5 \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\theta_x(t)\|^2 + \frac{1}{1080}\sum_{k=0}^{N-1}(\theta_{xxx,k})^2 h_k^5\right). \end{aligned} \tag{3.8}$$

Now we start estimating the terms on the right hand side of (3.7). For $\mathscr{I}_1$, we use the definition of the discrete innerproduct (2.5) and the Cauchy–Schwarz inequality followed by Young's inequality to obtain

$$\begin{aligned} |\mathscr{I}_1| &= |-\langle\eta_t, \theta_{xx}\rangle + \lambda\langle\eta, \theta_{xx}\rangle| \\ &\leqslant C(\varepsilon)(\|\eta_t\|_{L^\infty}^2 + \|\eta\|_{L^\infty}^2) + \varepsilon\|\theta_{xx}\|^2. \end{aligned} \tag{3.9}$$

To find an estimate for $\mathscr{I}_2$ and $\mathscr{I}_3$, we proceed as follows. Let $t^*\leqslant T$ be the largest time such that $u_h(x,t)$ exists for $0\leqslant t\leqslant t^*$ and let $\|\theta(t)\|_{L^\infty}\leqslant 1$ for all $t\leqslant t^*$. Since $\theta(0)=0$, $\|\theta(0)\|_{L^\infty}=0$, the existence of $t^*>0$ for which $u_h(x,t)$ exists can be inferred from Picard's existence theorem in ordinary differential equations.

For $0\leqslant t\leqslant t^*$, we can estimate $\mathscr{I}_2$ and $\mathscr{I}_3$ using the Cauchy–Schwarz inequality, Sobolev's inequality, Young's inequality and (3.3) as

$$|\mathscr{I}_2| = |\langle\theta\theta_x, \theta_{xx}\rangle| \leqslant C\|\theta(t)\|_{L^\infty}[\![\theta_x]\!][\![\theta_{xx}]\!] \leqslant C(\varepsilon)\|\theta_x\|^2 + \varepsilon\|\theta_{xx}\|^2, \tag{3.10}$$

and

$$|\mathscr{I}_3| = |\langle\theta\tilde{u}_{hx}, \theta_{xx}\rangle| \leqslant C(\varepsilon)\|\theta_x\|^2 + \varepsilon\|\theta_{xx}\|^2. \tag{3.11}$$

Similarly, we obtain the estimates for $\mathscr{I}_4$ and $\mathscr{I}_5$ as

$$|\mathscr{I}_4| = |\langle-\eta\tilde{u}_{hx}, \theta_{xx}\rangle| \leqslant C(\varepsilon)\|\eta\|_{L^\infty}^2 + \varepsilon\|\theta_{xx}\|^2, \tag{3.12}$$

and

$$|\mathscr{I}_5| = |\langle\tilde{u}_h\theta_x, \theta_{xx}\rangle| \leqslant C(\varepsilon)\|\theta_x\|^2 + \varepsilon\|\theta_{xx}\|^2. \tag{3.13}$$

Substituting (3.8)–(3.13) in (3.7) and combining the similar terms, we find that for $0 \leqslant t \leqslant t^*$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\theta_x(t)\|^2 + \frac{1}{1080}\sum_{k=0}^{N-1}(\theta_{xxx,k})^2 h_k^5\right) + v\|\theta_{xx}\|^2 \\ \leqslant C(\varepsilon)(\|\eta_t\|^2_{L^\infty} + \|\eta\|^2_{L^\infty} + \|\theta_x\|^2) + 5\varepsilon\|\theta_{xx}\|^2. \tag{3.14}$$

Integrating (3.14) with respect to time from 0 to $t \leqslant t^*$ and using the fact that $(1/1080)\sum_{k=0}^{N-1}(\theta_{xxx,k})^2 h_k^5 \geqslant 0$, we obtain

$$\|\theta_x(t)\|^2 + 2(v - 5\varepsilon)\int_0^t \|\theta_{xx}(\tau)\|^2\,\mathrm{d}\tau \leqslant C(\varepsilon)\int_0^t (\|\eta_t(\tau)\|^2_{L^\infty} + \|\eta(\tau)\|^2_{L^\infty})\,\mathrm{d}\tau \\ + C(\varepsilon)\int_0^t \|\theta_x(\tau)\|^2\,\mathrm{d}\tau. \tag{3.15}$$

Choosing $\varepsilon = v/10$ and using Lemma 3.1, (3.15) reduces to

$$\|\theta_x(t)\|^2 + v\int_0^t \|\theta_{xx}(\tau)\|^2\,\mathrm{d}\tau \leqslant Ch^8\int_0^T (\|u(\tau)\|^2_{W^{6,\infty}} + \|u_t(\tau)\|^2_{W^{6,\infty}})\,\mathrm{d}\tau \\ + C\int_0^t \|\theta_x(\tau)\|^2\,\mathrm{d}\tau. \tag{3.16}$$

An application of Gronwall's lemma now yields

$$\|\theta_x(t)\|^2 + v\int_0^t \|\theta_{xx}(\tau)\|^2\,\mathrm{d}\tau \leqslant Ch^8(\|u\|^2_{L^2(W^{6,\infty})} + \|u_t\|^2_{L^2(W^{6,\infty})}) \\ \forall t \in [0, t^*] \text{ with } t^* \leqslant T. \tag{3.17}$$

Since θ vanishes at the two end points, we easily obtain $\forall t \in [0, t^*]$ with $t^* \leqslant T$,

$$\|\theta(t)\|_{L^\infty} \leqslant \|\theta_x(t)\| \leqslant Ch^4(\|u\|_{L^2(W^{6,\infty})} + \|u_t\|_{L^2(W^{6,\infty})}). \tag{3.18}$$

Choose $h_0 > 0$ small enough so that for $h \in (0, h_0]$, $t \in [0, t^*]$, we have $\|\theta(t)\|_{L^\infty} \leqslant 1$.

If $t^* < T$, by the continuity of the map $t \mapsto \|\theta(t)\|_{L^\infty}$, we find that either $\|\theta(t)\|_{L^\infty} \leqslant 1\ \forall\ 0 \leqslant t \leqslant T$ or there exists some t^{**} such that $t^* < t^{**} \leqslant T$ and $\|\theta(t^{**})\|_{L^\infty} = 1$. In both the cases, we get a contradiction due to the fact that t^* is the largest time in $[0, T)$ such that $\|\theta(t)\|_{L^\infty} \leqslant 1$ for $t \in [0, t^*]$. Hence, $t^* = T$. Now a use of Lemma 3.1 along with triangle inequality completes the rest of the proof. □

Remark. As a consequence of (3.17) and Poincaré inequality, we obtain a superconvergence result for $\|\theta\|_{H^1}$.

Now, we discuss a priori error estimates for the error in $L^\infty(H^2)$- and $L^\infty(W^{1,p})$-norms.

Theorem 3.3. *Under the identical hypotheses of Theorem* 3.2, *there exists a constant* $C > 0$, *independent of* h, *such that for sufficiently small* h,

$$\|e\|_{L^\infty(H^2)} \leqslant Ch^2(\|u\|_{L^\infty(W^{6,2})} + \|u\|_{L^2(W^{6,\infty})} + \|u_t\|_{L^2(W^{6,\infty})}), \tag{3.19}$$

and for $1 \leqslant p \leqslant \infty$,

$$\|e(t)\|_{L^\infty(W^{1,p})} \leqslant Ch^3(\|u\|_{L^\infty(W^{6,2})} + \|u\|_{L^2(W^{6,\infty})} + \|u_t\|_{L^2(W^{6,\infty})}), \tag{3.20}$$

where $e = u - u_h$.

Proof. Setting $v_h = \theta_t$ in (3.4), it follows that

$$-\langle\theta_t, \theta_{txx}\rangle + v\langle\theta_{xx}, \theta_{txx}\rangle = -\langle\eta_t, \theta_{txx}\rangle + \lambda\langle\eta, \theta_{txx}\rangle + \langle\theta\theta_x, \theta_{txx}\rangle \\ + \langle\theta\tilde{u}_{hx}, \theta_{txx}\rangle - \langle\eta\tilde{u}_{hx}, \theta_{txx}\rangle + \langle\tilde{u}_h\theta_x, \theta_{txx}\rangle \\ = \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4 + \mathscr{I}_5 + \mathscr{I}_6. \tag{3.21}$$

Applying Lemma 2.2 to the first term on the left hand side of (3.21), we find that

$$-\langle\theta_t,\theta_{txx}\rangle \geqslant \|\theta_{tx}\|^2. \tag{3.22}$$

Integrating (3.22) with respect to time from 0 to t, we obtain

$$-\int_0^t \langle\theta_t,\theta_{txx}\rangle\,\mathrm{d}\tau \geqslant \int_0^t \|\theta_{tx}(\tau)\|^2\,\mathrm{d}\tau. \tag{3.23}$$

Similarly integrating the second term on the left hand side of (3.21) with respect to time, and using (2.12) we arrive at

$$\int_0^t \langle\theta_{xx},\theta_{txx}\rangle(\tau)\,\mathrm{d}\tau = \frac{1}{2}\|\theta_{xx}\|^2. \tag{3.24}$$

Now we start estimating the first term on the right hand side of (3.21). For $\mathscr{I}_1$, we rewrite it as

$$\mathscr{I}_1 = \langle -\eta_t,\theta_{txx}\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\langle\eta_t,\theta_{xx}\rangle + \langle\eta_{tt},\theta_{xx}\rangle. \tag{3.25}$$

Integrating (3.25) with respect to time from 0 to t and using Young's inequality along with the Cauchy–Schwarz inequality, we obtain

$$\left|\int_0^t \mathscr{I}_1(\tau)\,\mathrm{d}\tau\right| \leqslant C(\varepsilon)\left(\|\eta_t\|_{L^\infty}^2 + \int_0^t \|\eta_{tt}\|_{L^\infty}^2\,\mathrm{d}\tau + \int_0^t \|\theta_{xx}(\tau)\|^2\,\mathrm{d}\tau\right) + \varepsilon\|\theta_{xx}\|^2. \tag{3.26}$$

Estimating $\mathscr{I}_2$ by proceeding similarly as in the estimate for $\mathscr{I}_1$, we arrive at

$$\left|\int_0^t \mathscr{I}_2(\tau)\,\mathrm{d}\tau\right| \leqslant C(\varepsilon)\left(\|\eta\|_{L^\infty}^2 + \int_0^t (\|\eta_t(\tau)\|^2 + \|\theta_{xx}(\tau)\|^2)\,\mathrm{d}\tau\right) + \varepsilon\|\theta_{xx}\|^2. \tag{3.27}$$

To estimate $\mathscr{I}_3$, we rewrite it as

$$\mathscr{I}_3 = \langle\theta\theta_x,\theta_{txx}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle\theta\theta_x,\theta_{xx}\rangle - \langle\theta_t\theta_x + \theta\theta_{tx},\theta_{xx}\rangle \tag{3.28}$$

and hence, using the Sobolev embedding, Young's inequality and Theorem 3.2, we obtain

$$\left|\int_0^t \mathscr{I}_3(\tau)\,\mathrm{d}\tau\right| \leqslant C(\varepsilon)\left(\|\theta_x\|^2 + \int_0^t \|\theta_{xx}\|^2\,\mathrm{d}\tau\right) + \varepsilon\|\theta_{xx}\|^2 + \frac{1}{6}\int_0^t \|\theta_{tx}\|^2\,\mathrm{d}\tau. \tag{3.29}$$

Now, we rewrite $\mathscr{I}_4$ as

$$\mathscr{I}_4 = \langle\theta\tilde{u}_{hx},\theta_{txx}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle\theta\tilde{u}_{hx},\theta_{xx}\rangle - \langle\theta_t\tilde{u}_{hx} + \theta\tilde{u}_{hxt},\theta_{xx}\rangle \tag{3.30}$$

and hence

$$\begin{aligned}\left|\int_0^t \mathscr{I}_4(\tau)\,\mathrm{d}\tau\right| &\leqslant C(\varepsilon)\left(\|\theta_x\|^2 + \int_0^t (\|\theta_x(\tau)\|^2 + \|\theta_{xx}(\tau)\|^2)\,\mathrm{d}\tau\right)\\ &\quad + \frac{1}{6}\int_0^t \|\theta_{tx}\|^2\,\mathrm{d}\tau + \varepsilon\|\theta_{xx}\|^2.\end{aligned} \tag{3.31}$$

Similarly, we now obtain

$$\left|\int_0^t \mathscr{I}_5(\tau)\,\mathrm{d}\tau\right| \leqslant C(\varepsilon)\left(\|\eta\|_{L^\infty}^2 + \int_0^t (\|\eta_t\|_{L^\infty}^2 + \|\eta\|_{L^\infty}^2 + \|\theta_{xx}\|^2)\,\mathrm{d}\tau\right) + \varepsilon\|\theta_{xx}\|^2 \tag{3.32}$$

and

$$\left|\int_0^t \mathscr{I}_6(\tau)\,\mathrm{d}\tau\right| \leqslant C(\varepsilon)\left(\|\theta_x\|^2 + \int_0^t \|\theta_{xx}\|^2\,\mathrm{d}\tau\right) + \varepsilon\|\theta_{xx}\|^2 + \frac{1}{6}\int_0^t \|\theta_{xt}(\tau)\|^2\,\mathrm{d}\tau. \tag{3.33}$$

Substituting (3.23)–(3.33) in (3.21) and grouping the terms, we find that

$$\begin{aligned}\int_0^t \|\theta_{tx}(\tau)\|^2\,\mathrm{d}\tau + (v - 12\varepsilon)\|\theta_{xx}\|^2 \leqslant C(\varepsilon)[\|\eta_t\|^2_{L^\infty} + \|\eta\|^2_{L^\infty} + \|\theta\|^2_{L^\infty(H^1)}\\ + \int_0^t \left(\|\eta\|^2_{L^\infty} + \|\eta_t\|^2_{L^\infty} + \|\eta_{tt}\|^2_{L^\infty}\right)\mathrm{d}\tau\\ + \int_0^t (\|\theta_{xx}(\tau)\|^2 + \|\theta_x(\tau)\|^2)\,\mathrm{d}\tau].\end{aligned} \tag{3.34}$$

Choosing $\varepsilon = v/24$ in (3.34),

$$\begin{aligned}\int_0^t \|\theta_{tx}(\tau)\|^2\,\mathrm{d}\tau + \frac{v}{2}\|\theta_{xx}\|^2 \leqslant C(\varepsilon)[\|\eta_t\|^2_{L^\infty} + \|\eta\|^2_{L^\infty} + \|\theta\|^2_{L^\infty(H^1)}\\ + \int_0^t (\|\eta\|^2_{L^\infty} + \|\eta_t\|^2_{L^\infty} + \|\eta_{tt}\|^2_{L^\infty})\,\mathrm{d}\tau\\ + \int_0^t (\|\theta_x(\tau)\|^2 + \|\theta_{xx}(\tau)\|^2)\,\mathrm{d}\tau].\end{aligned} \tag{3.35}$$

Using Lemma 3.1, Theorem 3.2 and (3.17) we arrive at

$$\|\theta_{xx}\|^2 \leqslant C(v)h^8(\|u_t\|^2_{L^\infty(W^{6,\infty})} + \|u\|^2_{L^\infty(W^{6,\infty})}). \tag{3.36}$$

From Theorem 3.2 and the Poincaré inequality, we obtain the following superconvergence result for θ in H^2-norm:

$$\|\theta\|_{H^2} \leqslant Ch^4(\|u_t\|_{L^\infty(W^{6,\infty})} + \|u\|_{L^\infty(W^{6,\infty})}).$$

An application of the Sobolev imbedding theorem and the triangle inequality with Lemma 3.1 completes the rest of the proof. □

4. Numerical example

To test the qualocation method for Burgers' equation, the algorithm is used to solve the following initial–boundary value problem

$$u_t - vu_{xx} + uu_x = 0, \quad x \in (0,1),\ t \in (0,T], \tag{4.1}$$

$$u(x,0) = \sin(\pi x), \quad x \in [0,1], \tag{4.2}$$

$$u(0,t) = u(1,t) = 0, \quad t \in (0,T]. \tag{4.3}$$

The exact solution to this problem can be expressed as an infinite series [10],

$$u(x,t) = 2\pi v \frac{\sum_{m=1}^\infty a_m \mathrm{e}^{-m^2\pi^2 vt} m \sin(m\pi x)}{a_0 + \sum_{m=1}^\infty a_m \mathrm{e}^{-m^2\pi^2 vt}\cos(m\pi x)}, \tag{4.4}$$

where a_0, a_m, $m = 1, 2, \ldots$, are the Fourier coefficients defined, respectively, as

$$a_0 = \int_0^1 \mathrm{e}^{-(2\pi v)^{-1}[1-\cos(\pi x)]}\,\mathrm{d}x,$$

and $a_m = 2\int_0^1 \mathrm{e}^{-(2\pi v)^{-1}[1-\cos(\pi x)]}\cos(m\pi x)\,\mathrm{d}x.$

The sum $S_n(x,t) = 2\pi v\sum_{m=1}^n a_m \mathrm{e}^{-m^2\pi^2 vt} m\sin(m\pi x)/(a_0 + \sum_{m=1}^n a_m \mathrm{e}^{-m^2\pi^2 vt}\cos(m\pi x))$ is evaluated using Mathematica when $n = 11$ and is used as an approximation to the infinite sum (4.4). It has been observed that even if more

Table 1
Comparison of solution using qualocation method with the exact values for $\nu = 0.1$, $N = 10, 20$, $T = 0.1$

x	Numerical solution		Exact values
	$N = 10$	$N = 20$	
0.1	0.22345	0.22345	0.22345
0.2	0.43580	0.43580	0.43580
0.3	0.62512	0.62512	0.62512
0.4	0.77772	0.77772	0.77772
0.5	0.87728	0.87728	0.87728
0.6	0.90426	0.90425	0.90425
0.7	0.83695	0.83693	0.83692
0.8	0.65736	0.65731	0.65731
0.9	0.36580	0.36576	0.36575

than 11 terms i.e., $n = 12, 13, 14, \ldots$ are taken in evaluating the infinite sum (4.4), the values of the solution at the nodal points remain unchanged upto the precision of the computer. These values corrected up to 5 digits match with those reported in [1] as 'exact values' and are used in Table 1 for comparison.

Now, for applying the qualocation method, the interval $I = [0, 1]$ is decomposed into N uniform intervals each of width $h = 1/N$ and is extended on both sides as

$$x_{-3} \leqslant x_{-2} \leqslant x_{-1} \leqslant x_0 = 0 < x_1 < \cdots < x_N = 1 \leqslant \cdots \leqslant x_{N+3}.$$

Using C^2- cubic B-splines, the approximate solution $u_h \in S_h^0$ can be expressed as

$$u_h(x, t) = \sum_{k=-1}^{N+1} \gamma_k(t) B_k(x), \quad x \in [0, 1], \quad t \in [0, T], \tag{4.5}$$

where B_k is the cubic B-spline with support in $[x_{k-2}, x_{k+2}]$ defined by

$$B_k(x) = \frac{1}{h^3} \begin{cases} (x - x_{k-2})^3 & \text{if } x \in [x_{k-2}, x_{k-1}], \\ h^3 + 3h^2(x - x_{k-1}) + 3h(x - x_{k-1})^2 - 3(x - x_{k-1})^3 & \text{if } x \in [x_{k-1}, x_k], \\ h^3 + 3h^2(x_{k+1} - x) + 3h(x_{k+1} - x)^2 - 3(x_{k+1} - x)^3 & \text{if } x \in [x_k, x_{k+1}], \\ (x_{k+2} - x)^3 & \text{if } x \in [x_{k+1}, x_{k+2}], \\ 0 & \text{otherwise.} \end{cases}$$

Using (4.3), we obtain $\sum_{k=-1}^{N+1} \gamma_k(t) B_k(0) = \sum_{k=-1}^{N+1} \gamma_k(t) B_k(1) = 0$.

Note that,

$$B_k(x_j) = \begin{cases} 4 & \text{if } j = k, \\ 1 & \text{if } j = k - 1 \text{ or } j = k + 1, \\ 0 & \text{if } j = i + 1 \text{ or } j = i - 1, \end{cases}$$

and $B_k(x) \equiv 0$ for $x \geqslant x_{k+2}$ and $x \leqslant x_{k-2}$.

Also for each m, $m = 0, 1, \ldots, N$, let v_m denote the piecewise linear hat function at each of the nodal points x_m, $m = 0, \ldots, N$ such that

$$v_m(x) = \begin{cases} \dfrac{x - x_{m-1}}{h}, & x \in [x_{m-1}, x_m], \\ \dfrac{x_{m+1} - x}{h}, & x \in [x_m, x_{m+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Multiply (4.1) by the test functions v_m, $m = 0, \ldots, N$, and integrate with respect to space variable between [0, 1]. Integrals are approximated by using the two-point Gauss quadrature rule (2.5). Also replacing the expressions for γ_{-1}

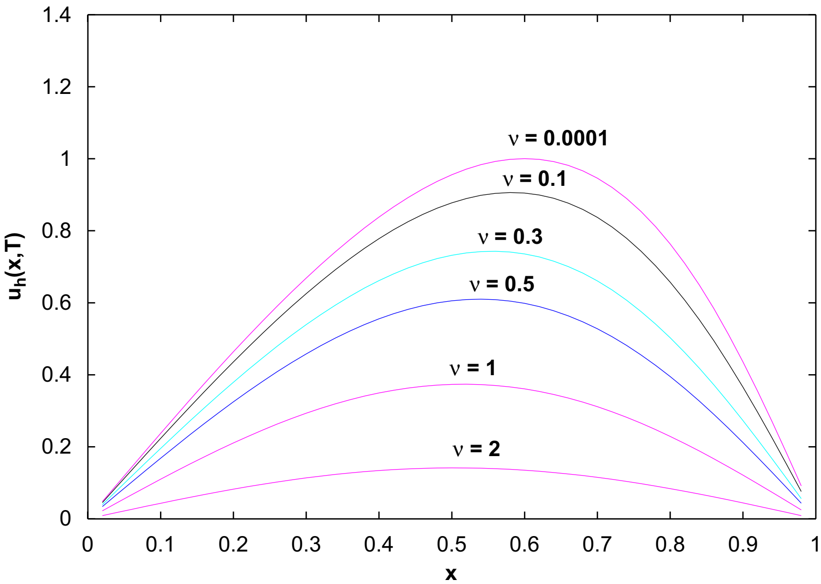

Fig. 1. $u_h(x, 0.1)$ for $\nu = 0.0001, 0.1, 0.3, 0.5, 1, 2$.

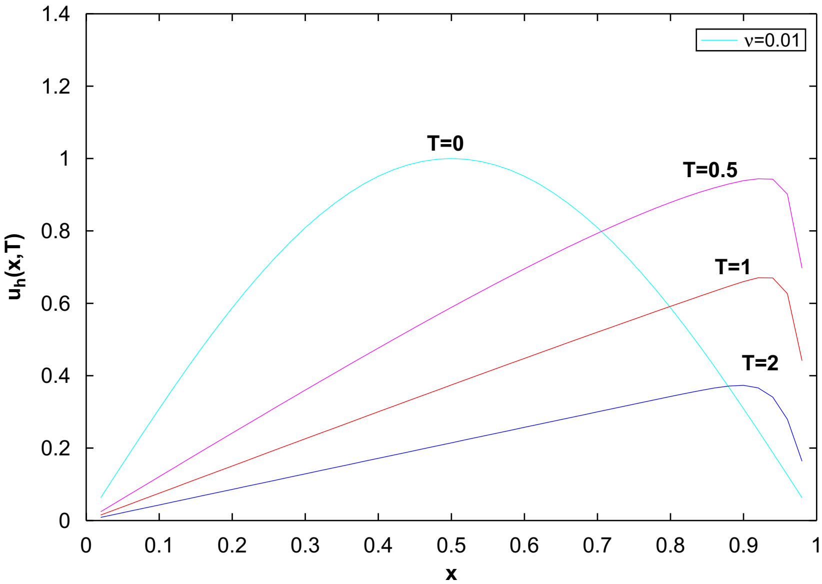

Fig. 2. $u_h(x, T)$ for $T = 0, 0.5, 1, 2$; $\nu = 0.01$.

and γ_{N+1} using boundary conditions, a system of $N + 1$ nonlinear ordinary differential equations in $N + 1$ unknowns is obtained. Using the extrapolated Crank–Nicolson method (see [16,19]), we obtain a pentadiagonal system of linear algebraic equations at each time step. On solving this system, we obtain the values of $\gamma_1, \ldots, \gamma_{N+1}$.

In Table 1, a comparison of the numerical solutions with the exact solution for $\nu = 0.1$, $N = 10, 20$ and $T = 0.1$ at different points of $(0, 1)$ are shown. Fig. 1 shows the profiles of the approximate solutions for a fixed value of $T = 0.1$ and for different values of ν.

In Fig. 2, the profiles of the approximate solutions for the fixed value of $\nu = 0.01$ and for different values of T have been given. This figure shows that for $\nu = 0.01$ and for different values of $T = 0, 0.5, 1, 2$ the maximum point of the solution tilts towards the right end point.

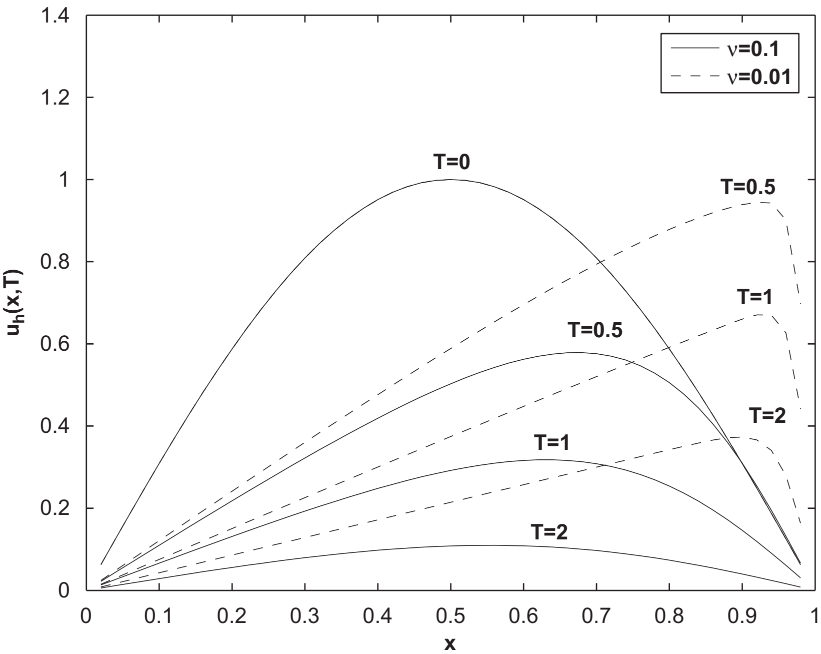

Fig. 3. $u_h(x, T)$ for $T = 0, 0.5, 1, 2, 3$; $v = 0.1, 0.01$.

Table 2
Order of convergence of the solution u_h and its derivative in L^∞ and $W^{1,\infty}$-norm, respectively, for $v = 0.1$, $T = 1$

$h = \frac{1}{N}$	L^∞-norm		$W^{1,\infty}$-norm	
	$E = \|u - u_h\|_{L^\infty}$	Order	$E = \|u - u_h\|_{W^{1,\infty}}$	Order
$\frac{1}{4}$	0.00436390678186		0.06185445247522	
$\frac{1}{8}$	1.723783074021323e-04	4.66197014640545	0.00548829852578	3.49444658713220
$\frac{1}{16}$	9.287197316687479e-06	4.21419113571726	7.489586720055286e-04	2.87340093915138
$\frac{1}{32}$	4.797979710480238e-07	4.27474432493959	8.252222018323430e-05	3.18203157087031
$\frac{1}{64}$	2.964515430359249e-08	4.01655884571358	9.955938154115884e-06	3.05115346261239
$\frac{1}{128}$	1.839708024808928e-09	4.01024758275134	1.234645049530414e-06	3.01146093012464

In Fig. 3, the profiles for $v = 0.01$ and $v = 0.1$ and for different values of T are compared. We observe that the propagation front is steeper for smaller values of viscosity.

In Table 2, computation of the order of convergence of solution u_h and its derivative (when $v=0.1, T=1$) in L^∞-norm and $W^{1,\infty}$-norm, respectively, has been shown. Fig. 4 represents the graph of error $E = \|u - u_h\|_\infty$ as a function of the discretization step h in the log-log scale when $T = 0.1$, and $v = 0.1$. It is shown that the slope is approximately 4 confirming the theoretical order of convergence for $v = 0.1$. Note that the computational order of convergence agrees with the theoretical order of convergence in Table 2 and also in Fig. 4.

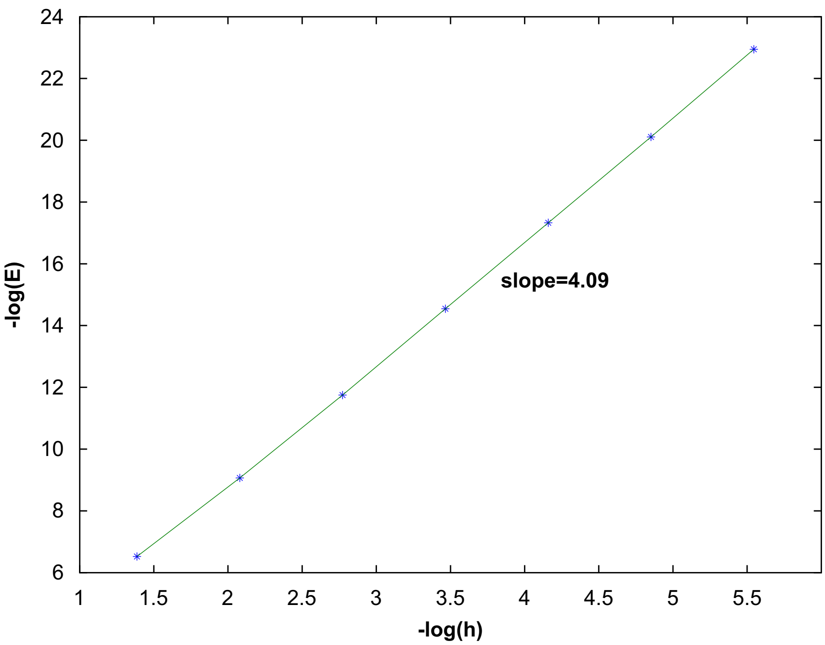

Fig. 4. Relative error E as a function of discretization step h in the log–log scale.

5. Conclusions

A numerical scheme for the solution of Burgers' equation based on a qualocation method is discussed. Discretization is achieved by approximating the integrals by composite two-point Gauss quadrature rule. Under suitable assumptions the solution is shown to have an $O(h^{4-i})$ rate of convergence in $W^{i,p}$ norm for $i = 0, 1, 1 \leqslant p \leqslant \infty$ and order $O(h^2)$ in H^2 norm. The scheme has been implemented successfully. The numerical example solved using qualocation method supports the theoretical result obtained in this paper.

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